

# Feynman Rules for Piecewise Linear Wilson Lines

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## Outline

- 1 Linear Wilson Lines
- 2 Piecewise Wilson Lines
- 3 Piecewise Linear Wilson Lines: Methodology
- 4 Example Calculation

# Path-Ordered Exponentials

## Wilson Line

$$\begin{aligned}\mathcal{U}[\mathcal{C}] &= \mathcal{P} \exp \left( -i g \int_{\mathcal{C}} dz^{\mu} A_{\mu}(z) \right) \\ &= \mathcal{P} \exp \left( -i g \int_a^b d\lambda (z^{\mu})' A_{\mu}(\lambda) \right)\end{aligned}$$

# Path-Ordered Exponentials

## Path-Ordering for linear lines

$$z^\mu = r^\mu + \hat{n}^\mu \lambda \quad \lambda = a \dots b$$

$$\frac{1}{m!} \int_c \cdots \int_c dz_1 \cdots dz_m = \int_a^b \int_{\lambda_1}^b \cdots \int_{\lambda_{m-1}}^b d\lambda_1 \cdots d\lambda_m$$

$$= \int_a^b \int_a^{\lambda_m} \cdots \int_a^{\lambda_2} d\lambda_m \cdots d\lambda_1$$

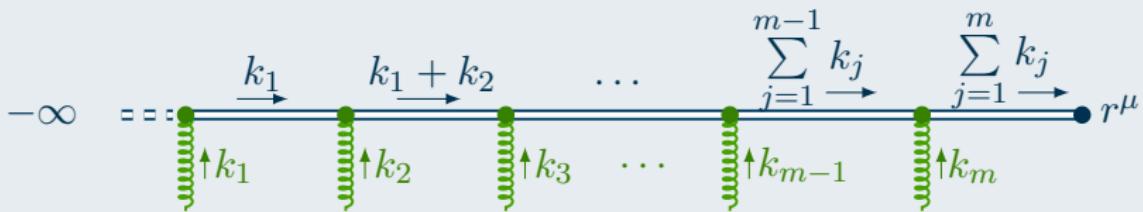
# Feynman Rules

## Wilson Line Bounded From Above

$$\begin{aligned} \mathcal{U}_{(r; -\infty)} &= \sum_{m=0}^{\infty} (-ig)^m \int \left( \frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(k_m) \cdots \hat{n} \cdot A(k_1) \times \cdots \\ K(j) &= \sum_{l=1}^j k_l \end{aligned}$$

$$\cdots \times e^{-ir \cdot K} \prod_{j=1}^m \frac{i}{\hat{n} \cdot K(j) + i\eta}$$

## Feynman Diagram



# Feynman Rules

## Feynman Rules for Linear Wilson Lines

1) Wilson line propagator:

$$\overbrace{\text{---}}^k = \frac{i}{\hat{n} \cdot k + i\eta}$$

2) external point:

$$\overbrace{\text{---}}^k \bullet r^\mu = e^{-ir \cdot k}$$

3) infinite point:

$$\dots\dots\dots +\infty = 1 \quad (k=0)$$

4) Wilson vertex:

$$j \overbrace{\text{---}}^{\mu, a} i = -ig \hat{n}^\mu (t^a)_{ij}$$

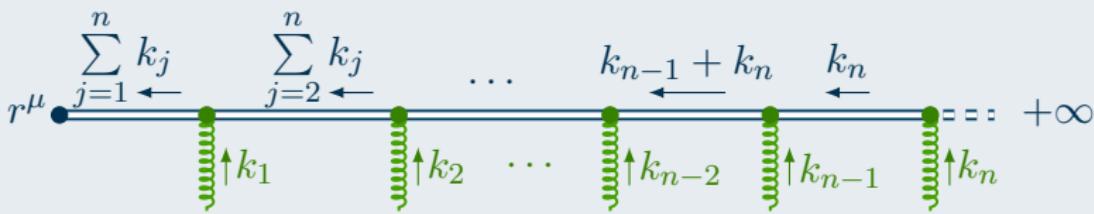
# Feynman Rules

## Wilson Line Bounded From Below

$$\mathcal{U}_{(+\infty; r)} = \sum_{m=0}^{\infty} (-i g)^m \int \left( \frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(k_m) \cdots \hat{n} \cdot A(k_1) \times \cdots$$

$$\tilde{K}(j) = \sum_{l=1}^j k_{m-l+1} \quad \cdots \times e^{-ir \cdot K} \prod_{j=1}^m \frac{-i}{\hat{n} \cdot \tilde{K}(j) - i\eta}$$

## Feynman Diagram



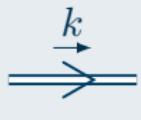
# Feynman Rules

## Reversals

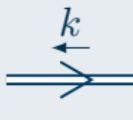
$$\begin{array}{ccc} \overrightarrow{\text{---}}^k & = & \frac{i}{\hat{n} \cdot k + i\eta} \\ & & \end{array} \quad \begin{array}{ccc} \overleftarrow{\text{---}}^k & = & \frac{-i}{\hat{n} \cdot k - i\eta} \end{array}$$

# Feynman Rules

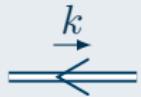
## Reversals



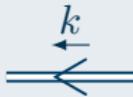
$$= \frac{i}{\hat{n} \cdot k + i\eta}$$



$$= \frac{-i}{\hat{n} \cdot k - i\eta}$$



$$= \frac{-i}{\hat{n} \cdot k - i\eta}$$



$$= \frac{i}{\hat{n} \cdot k + i\eta}$$

$$j \xrightarrow[\mu, a]{} i$$

$$= -ig \hat{n}^\mu (t^a)_{ij}$$

$$j \xleftarrow[\mu, a]{} i$$

$$= ig \hat{n}^\mu (t^a)_{ij}$$

## More Types

## Finite Line

$$\mathcal{U}_{(b; a)} = \sum_{n=0}^{\infty} (-i g)^m \int \left( \frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(k_m) \cdots \hat{n} \cdot A(k_1) \times \cdots \\ \cdots \times \sum_{l=0}^m e^{-ia \cdot K(l)} e^{-ib \cdot K(m-l)} \prod_{j=1}^l \frac{-i}{\hat{n} \cdot \tilde{K}(j)} \prod_{j=l+1}^m \frac{i}{n \cdot K(j)}$$

## Hermitian Conjugate

$$\mathcal{U}_{(r; -\infty)}^\dagger = \sum_{n=0}^{\infty} (ig)^m \int \left( \frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(k_1) \cdots \hat{n} \cdot A(k_m) \times \cdots \\ \cdots \times e^{-ib \cdot K} \prod_{j=1}^m \frac{i}{\hat{n} \cdot K(j) + i\eta}$$

## More Types

## Finite Lines and Hermitian Conjugates

$$a^\mu \quad b^\mu = \quad \text{Diagram: two parallel horizontal lines with arrows pointing right, each ending in a black dot. A tensor product symbol (\otimes) is placed between them.}$$

$$(\Rightarrow\bullet)^\dagger = \quad \text{Diagram: a single horizontal line with an arrow pointing left, ending in a black dot.} \quad (\bullet\Rightarrow)^\dagger = \quad \text{Diagram: a single horizontal line with an arrow pointing right, ending in a black dot.}$$

# Different Types

## Semi-Infinite Lines and Path Reversals

$$\bullet \overrightarrow{\hspace{1cm}} \hat{n}^\mu (-ig)^m A_m \cdots A_1 e^{-ir \cdot K} \prod_j^m \frac{-i}{\hat{n} \cdot \tilde{K}(j) - i\eta} \stackrel{N}{=} A^m(r, \hat{n})$$

$$\overleftarrow{\hspace{1cm}} \bullet (ig)^m A_m \cdots A_1 e^{-ir \cdot K} \prod_j^m \frac{-i}{\hat{n} \cdot \tilde{K}(j) - i\eta} \stackrel{N}{=} B^m(r, \hat{n})$$

$$\bullet \overleftarrow{\hspace{1cm}} (ig)^m A_m \cdots A_1 e^{-ir \cdot K} \prod_j^m \frac{i}{\hat{n} \cdot \tilde{K}(j) + i\eta} = A^m(r, -\hat{n})$$

$$\overrightarrow{\hspace{1cm}} \bullet (-ig)^m A_m \cdots A_1 e^{-ir \cdot K} \prod_j^m \frac{i}{\hat{n} \cdot \tilde{K}(j) + i\eta} = B^m(r, -\hat{n})$$

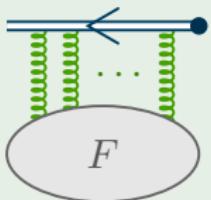
Relation Between  $A^m$  and  $B^m$ Relation Between  $A^m$  and  $B^m$ 

$$B^m(r, \hat{n}) = (-)^m A^m(r, \hat{n}) \Big|_{k_1 \rightarrow k_n, \dots, k_n \rightarrow k_1}$$



Relation Between  $A^m$  and  $B^m$ Relation Between  $A^m$  and  $B^m$ 

$$B^m(r, \hat{n}) = (-)^m A^m(r, \hat{n}) \Big|_{k_1 \rightarrow k_n, \dots, k_n \rightarrow k_1}$$



$$= (-)^m \int \left( \frac{dk_i}{16\pi^4} \right)^m A^m(r, \hat{n}) F_{a_1 \dots a_m}^{\mu_1 \dots \mu_m}(k_m, \dots, k_1)$$

(absorb gluon propagators in  $F$ )

# Relation Between $A^m$ and $B^m$

## Symmetrising the Blob

- symmetrise  $F$  simultaneously in  $k_i$ ,  $\mu_i$  and  $a_i$   
(identical to making all crossings)
- because all Lorentz indices are contracted with the same  $\hat{n}^\mu$ ,  
 $F$  is automatically symmetric in  $\mu_i$
- interchanging  $k_i$  and  $k_j$  is thus same as interchanging  $a_i$  and  $a_j$
- sometimes  $F$  will have straightforward color symmetry

Relation Between  $A^m$  and  $B^m$ 

## Symmetrising the Blob

$$\text{Diagram showing the symmetrisation of a blob } F. \text{ On the left, a blob } F \text{ is connected to two horizontal lines by vertical gluon lines. An arrow points from the left line to the right line. On the right, the blob is shown with an arrow pointing from the right line to the left line, labeled with } (-)^m \text{ and } F|_{\overline{\{a_i\}}}. \text{ Ellipses indicate continuation.}$$

## Easy Blob Example: 3-Gluon Vertex

$$\text{Diagram showing the symmetrisation of a 3-gluon vertex. On the left, a 3-gluon vertex is connected to three horizontal lines by gluon lines. An arrow points from the top-left line to the top-right line. On the right, the 3-gluon vertex is shown with an arrow pointing from the top-right line to the top-left line. Ellipses indicate continuation.}$$



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# Piecewise Path-Ordered Exponentials

## Wilson Line With $M$ Segments

$$\mathcal{U}(\lambda) = \begin{cases} \mathcal{U}^A(\lambda) & \lambda = a_1 \dots a_2 \\ \mathcal{U}^B(\lambda) & \lambda = a_2 \dots a_3 \\ \vdots \\ \mathcal{U}^M(\lambda) & \lambda = a_M \dots a_{M+1} \end{cases}$$

## Result for Full Wilson Line

$$\mathcal{U}_1 = \sum_{J=1}^M \mathcal{U}_1^J$$

$$\mathcal{U}_2 = \sum_{J=1}^M \mathcal{U}_2^J + \sum_{K=2}^M \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J$$

# Piecewise Path-Ordered Exponentials

## Result for Full Wilson Line

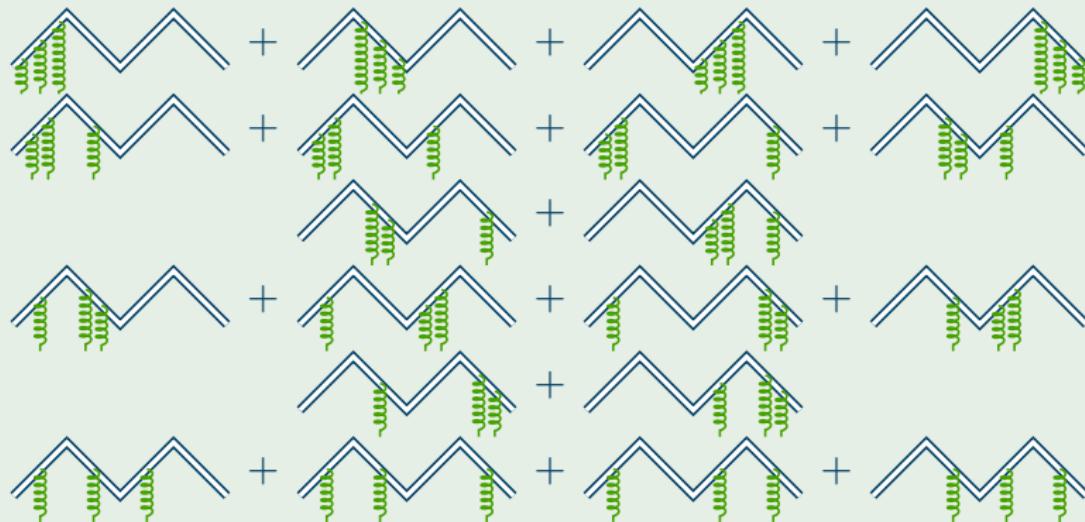
$$\mathcal{U}_3 = \sum_{J=1}^M \mathcal{U}_3^J + \sum_{J=2}^M \sum_{K=1}^{J-1} [\mathcal{U}_1^J \mathcal{U}_2^K + \mathcal{U}_2^J \mathcal{U}_1^K] + \sum_{J=3}^M \sum_{K=2}^{J-1} \sum_{L=1}^{K-1} \mathcal{U}_1^J \mathcal{U}_1^K \mathcal{U}_1^L$$

⋮

$$\mathcal{U}_m = \sum_{i=1}^m \left[ \left( \prod_{j=1}^i \sum_{J_j=i-j+1}^{J_{j-1}-1} \right)_{J_0-1=M} \left( \begin{array}{l} \text{All terms of the form } \prod_{j=1}^i \mathcal{U}_{l_j}^{J_j} \\ \text{such that } \sum_{j=1}^i l_j = m \end{array} \right) \right]$$

# Piecewise Path-Ordered Exponentials

Illustration for  $\mathcal{U}_3$



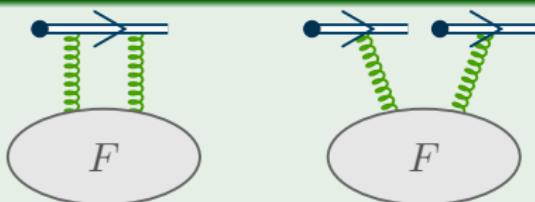


## Outline

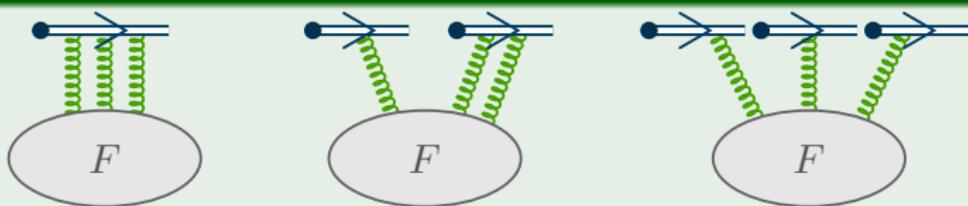
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# Basic Diagrams

$m = 2$

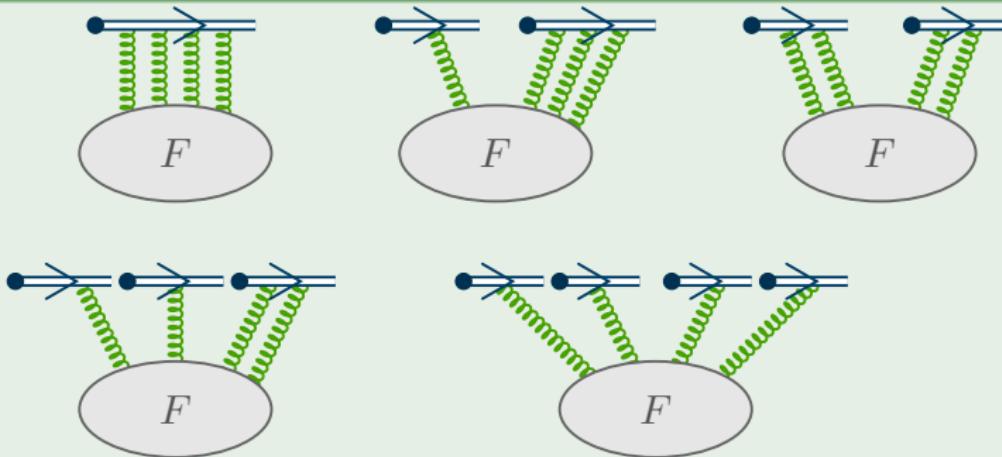


$m = 3$



# Basic Diagrams

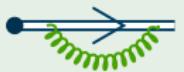
$m = 4$



## Blob Examples

 $m = 2$ 

$$= \delta^{ab} \delta^{(4)}(k_1 - k_2) D_{\mu\nu}(k_1)$$



$$= \quad \quad \quad$$

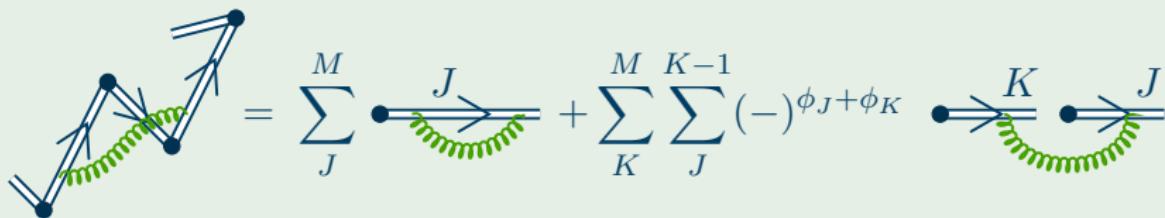


$$= \quad \quad \quad = - \quad \quad \quad$$

## Blob Examples

 $m = 2$ 

$$\mathcal{U}_2 = \sum_{J=1}^M \mathcal{U}_2^J + \sum_{K=2}^M \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J$$



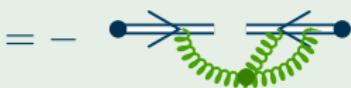
$$\phi_J = \begin{cases} 0 & \text{---} \\ 1 & \text{--\textbf{\textarrowright}} \end{cases}$$

# Blob Examples

$m = 3$



$$= g f^{abc} D_{\mu_1 \nu_1}^{k_1} D_{\mu_2 \nu_2}^{k_2} D_{\mu_3 \nu_3}^{k_3} g^{\nu_1 \nu_2} (k_1 - k_2)^{\nu_3} + \text{cross.}$$



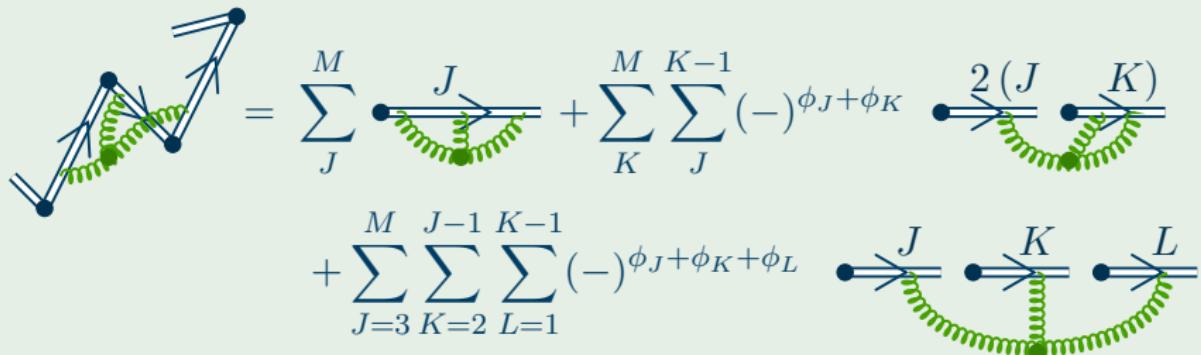
= -

etc.

## Blob Examples

 $m = 3$ 

$$\mathcal{U}_3 = \sum_{J=1}^M \mathcal{U}_3^J + \sum_{J=2}^M \sum_{K=1}^{J-1} [\mathcal{U}_1^J \mathcal{U}_2^K + \mathcal{U}_2^J \mathcal{U}_1^K] + \sum_{J=3}^M \sum_{K=2}^{J-1} \sum_{L=1}^{K-1} \mathcal{U}_1^J \mathcal{U}_1^K \mathcal{U}_1^L$$



The diagram illustrates the decomposition of a complex 3-loop blob (left) into simpler components. It is shown as a sum of two terms:

$$= \sum_J^M \text{ (1-loop diagram with J)} + \sum_K^M \sum_{J=1}^{K-1} (-)^{\phi_J + \phi_K} \text{ (2-loop diagram with J, K)} + \sum_{J=3}^M \sum_{K=2}^{J-1} \sum_{L=1}^{K-1} (-)^{\phi_J + \phi_K + \phi_L} \text{ (3-loop diagram with J, K, L)}$$

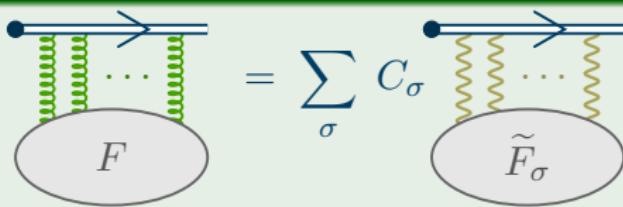
The 1-loop diagram consists of a single horizontal line with a green wavy loop attached. The 2-loop diagram consists of two horizontal lines connected by a green wavy loop. The 3-loop diagram consists of three horizontal lines connected by a green wavy loop.

# Non-Trivial Color Structure

## Blob With Non-Trivial Color Structure

$$F_{\mu_1 \dots \mu_m}^{a_1 \dots a_m}(k_1, \dots, k_m) = \sum_{\text{perm}} C^{\sigma(a_1 \dots a_m)} \tilde{F}_{\sigma(\mu_1 \dots \mu_m)} (\sigma(k_1, \dots, k_m))$$

## Factorize Out Color



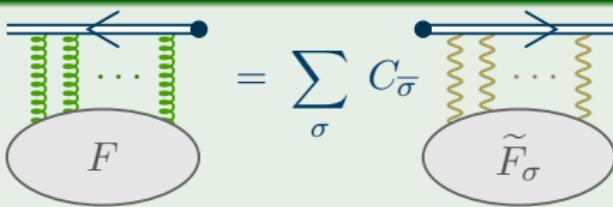
$$C_\sigma = t^{a_n} \dots t^{a_1} C^{\sigma(a_1 \dots a_m)}$$

# Non-Trivial Color Structure

## Blob With Non-Trivial Color Structure

$$F_{\mu_1 \dots \mu_m}^{a_1 \dots a_m}(k_1, \dots, k_m) = \sum_{\text{perm}} C^{\sigma(a_1 \dots a_m)} \tilde{F}_{\sigma(\mu_1 \dots \mu_m)} (\sigma(k_1, \dots, k_m))$$

## Factorize Out Color



$$C_{\bar{\sigma}} = t^{a_1} \dots t^{a_n} C^{\sigma(a_1 \dots a_m)}$$

# Blob Example With Non-Trivial Color Structure

$m = 4$


$$= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \tilde{F}_{\mu_1 \dots \mu_4}(k_1, \dots, k_4)$$

# Blob Example With Non-Trivial Color Structure

$m = 4$

$$\begin{array}{c} \text{Diagram: A green wavy line connecting two horizontal lines with arrows pointing right.} \\ = \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \tilde{F}_{\mu_1 \dots \mu_4}(k_1, \dots, k_4) \\ \text{Diagram: Two horizontal lines with arrows pointing right, connected by a green wavy line.} \end{array}$$

# Blob Example With Non-Trivial Color Structure

$m = 4$

$$\begin{aligned} \text{Diagram 1} &= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \tilde{F}_{\mu_1 \dots \mu_4}(k_1, \dots, k_4) \\ \text{Diagram 2} &= \text{Diagram 3} \\ \text{Diagram 4} &= C_1 \text{ (blob)} \tilde{F}_{1234} + C_2 \text{ (blob)} \tilde{F}_{1324} + C_3 \text{ (blob)} \tilde{F}_{1423} \end{aligned}$$

The diagrams consist of two horizontal blue lines with arrows indicating direction. Green wavy lines connect them. The first diagram shows a single green wavy line connecting both lines. The second diagram shows a green wavy line connecting the top line to a point on the bottom line. The third diagram shows a green wavy line connecting the bottom line to a point on the top line. The fourth diagram shows two green wavy lines, one connecting the top line to the left of the other, which then connects to the bottom line.

# Blob Example With Non-Trivial Color Structure

$m = 4$

$$\text{Diagram} = \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \tilde{F}_{\mu_1 \dots \mu_4}(k_1, \dots, k_4)$$

$$\text{Diagram} = \text{Diagram}$$

$$\text{Diagram} = C_1 \tilde{F}_{1234} + C_2 \tilde{F}_{1324} + C_3 \tilde{F}_{1423}$$

$$\text{Diagram} = -C_3 \tilde{F}_{1234} - C_2 \tilde{F}_{1324} - C_1 \tilde{F}_{1423}$$

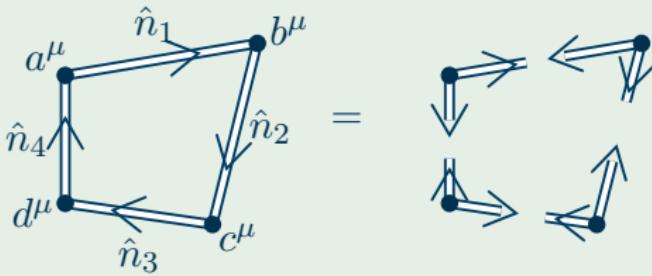


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# Quadrilateral Wilson Loop

## Quadrilateral Wilson Loop



# Quadrilateral Wilson Loop

## First Order


$$= F(r_J, \hat{n}_J)$$
  

$$= G(r_J, r_K, \hat{n}_J, \hat{n}_K)$$

# Quadrilateral Wilson Loop

## First Order

$$\begin{aligned}\mathcal{U}_2 &= \sum_{J=1}^M \mathcal{U}_2^J + \sum_{K=2}^M \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J \\ &= \sum_J^M F(r_J, \hat{n}_J) + \sum_{K=2}^M \sum_{J=1}^{K-1} (-)^{J+K} G(r_J, r_K, \hat{n}_J, \hat{n}_K)\end{aligned}$$

## Result for Light-Like Loop

$$\begin{aligned}\mathcal{U}_2 &= \frac{\alpha_s C_F}{\pi} (-2\pi\mu^2)^\epsilon \Gamma(1-\epsilon) \times \dots \\ &\quad \dots \times \left[ \frac{1}{\epsilon^2} \left( (b-d)^2 - i\eta \right)^\epsilon + \frac{1}{\epsilon^2} \left( (c-a)^2 - i\eta \right)^\epsilon \right]\end{aligned}$$

# Conjecture

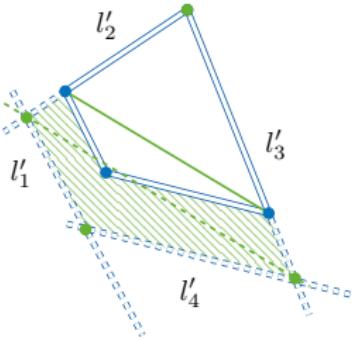
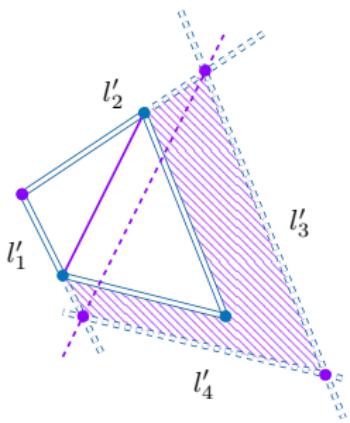
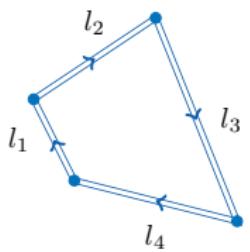
## Geometric Evolution of Light-Like Quadrilateral

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \left\langle \frac{\delta}{\delta \ln \Sigma} \right\rangle \ln \mathcal{W}_\gamma = - \sum_{\text{cusps}} \Gamma_{\text{cusp}}$$

Gamma cusp at NLO:

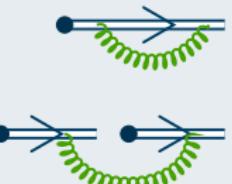
$$\Gamma_{\text{cusp}}(g) = \frac{\alpha_s}{\pi} C_F + \left( \frac{\alpha_s}{\pi} \right)^2 C_F \left( C_A \left( \frac{67}{36} - \frac{\pi^2}{12} \right) - N_f \frac{5}{18} \right)$$

# Conjecture



# Reusability

## Example


$$\begin{aligned} &= \mathrm{i}g^2 \frac{C_F}{16\pi} \frac{\hat{n}^2}{\eta} (-4\pi\mu^2)^\epsilon \Gamma(\epsilon) X(\hat{n}^2, \epsilon) \\ &= g^2 \frac{C_F}{16\pi} \frac{\hat{n}_1 \cdot \hat{n}_2}{\hat{n}_1 \sqrt{\tilde{n}_2 \cdot \hat{n}_2}} \left( 2\pi \mathrm{i} \mu^2 \frac{1}{\eta} \sqrt{\frac{R}{N_2}} \right)^\epsilon Y_\epsilon(\mathrm{i}\eta\sqrt{RN_2}) \end{aligned}$$

# Conclusions

## Conclusions & Outlook

- framework to minimize number of diagrams for piecewise linear Wilson lines
  - lesser diagrams in exchange for more general (and thus more complicated) integrals
  - only interesting for  $M > 2$
- 
- calculate higher orders
  - include final-state cut
  - try framework for TMD Wilson line structure